Cork Institute of Technology

Special Mathematics Examination for Engineering Degree Entry

June 2006

Time: 2 hours and 30 minutes

PAPER 2 (300 marks)

Attempt **FIVE** questions from Section **A** and **ONE** question from Section **B**.
Each question carries 50 marks.

WARNING: Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

SECTION A

Answer FIVE questions from this section

1. (a) The centre of the circle T is $(0, 0)$. $(-1, 4)$ is a point on T.

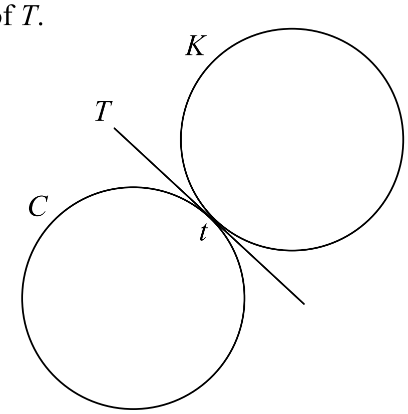
(i) Find the equation of T.

(ii) Write down the co-ordinates of three other points of T.

(b) The circle C has centre $(2, 2)$ and the circle K has centre $(4, 4)$.

The point t is on C and K is the image of C under the central symmetry in t.

The line T is the common tangent to C and K at t.



(i) Find the equation of C.

(ii) Find the equation of the tangent T.

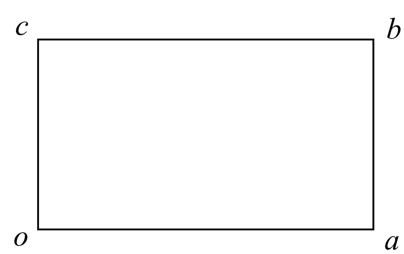
(iii) Give a reason why the circle C does not intersect the x-axis or the y-axis.

(c) The circle S has equation $x^2 + y^2 - 4x + 6y - 12 = 0$.

(i) Verify that $(-2, 0)$ is on S.

(ii) Find the equation of the tangent to S at the point $(-2, 0)$.

2. (a) Copy the rectangle $oabc$, where o is the origin, into your answer book. On your diagram construct



(i) the point p, where $\vec{p} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$

(ii) the point q, where $\vec{q} = \vec{c} - \vec{b}$.

(b) $\vec{p} = 4\vec{i} + 5\vec{j}$ and $\vec{q} = 2\vec{i} + 8\vec{j}$.

(i) Calculate $|\overrightarrow{pq}|$.

(ii) If $h\vec{p} + k\vec{q} = 2\vec{i} - 14\vec{j}$, where $h, k \in \mathbf{R}$, find the value of h and the value of k.

(c) $\vec{u} = -3\vec{i} + \vec{j}$ and $\vec{v} = -2\vec{i} - \vec{j}$.

(i) Show that $\vec{u}.\vec{u} = |\vec{u}|^2$.

(ii) Find $|\angle uov|$, where o is the origin.

(iii) If $\vec{w} = h\vec{i} - 2\vec{j}$ and $\vec{u} \perp \vec{w}$, find $h \in \mathbf{R}$.

3. (a) L is the line $mx + y - 2m = 0$. The point $(3, -4)$ is on L. Find m.

(b) $a(-1, 3)$ and $c(8, -9)$ are two points.

b is a point on $[ac]$ such that $|ab|:|bc| = 2:1$.

(i) Find the co-ordinates of b.

(ii) d is a point on the line $y - 3 = 0$ such that $|ab| = |bd|$.
Find the co-ordinates of d.

(iii) Find the area of the triangle abd.

(iv) f is the transformation $(x, y) \to (x', y')$ where $x' = x + y$ and $y' = 2x - y$.
Find the co-ordinates of $f(a)$, $f(b)$ and $f(c)$.

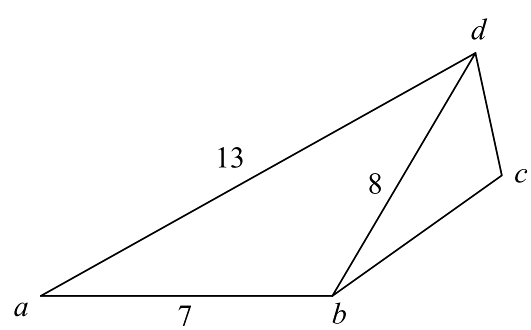
(v) Investigate if $|f(a)f(b)|:|f(b)f(c)| = |ab|:|bc|$.

4. (a) Express $\tan 75°$ in the form $p + \sqrt{q}$ where $p, q \in \mathbf{N}$.

(b) $abcd$ is a quadrilateral in which
$|ab| = 7$, $|bd| = 8$ and $|ad| = 13$.

(i) Show that $|\angle abd| = 120°$.

(ii) Given that the quadrilateral $abcd$ has area $\frac{35\sqrt{3}}{2}$, find the ratio area Δabd : area Δbcd.



(c) A triangle has sides a, b and c.

The angles opposite a, b and c are A, B and C, respectively.

(i) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

(ii) Hence, or otherwise, prove that

$$\frac{1}{c\cos B - b\cos C} = \frac{a}{c^2 - b^2}.$$

5. (a) Solve $\cos A = -0.5, \quad 0° \le A \le 360°$.

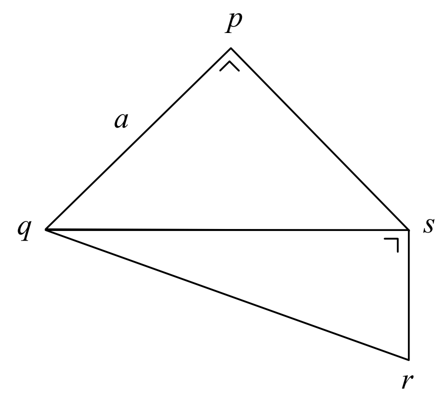
(b) **(i)** Using $\cos 2A = \cos^2 A - \sin^2 A$, or otherwise, prove that

$$1 - \cos 2A = 2\sin^2 A.$$

(ii) Hence, or otherwise, prove that

$$\frac{\sin 2A - \cos 2A + 1}{\sin 2A + \cos 2A - 1} = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

(c) *pqrs* is a quadrilateral in which
$|pq| = a, \quad |\angle pqs| = 45°, \quad |\angle sqr| = 30°,$
$pq \perp ps$ and $qs \perp sr$.



(i) Write $|qs|$ in terms of a.

(ii) Write the perimeter of the triangle *qrs* in the form $\left(\sqrt{h} + \sqrt{k}\right)a$, where $h, k \in \mathbf{N}$.

6. (a) A mathematics paper is in two sections. Section A contains 7 questions and section B contains 4 questions.
A student is required to answer 5 questions from section A and 1 question from section B.
How many different choices of questions does the student have?

(b) Solve the difference equation

$$u_{n+2} + 3u_{n+1} - 10u_n = 0, \quad n \ge 0$$

given that $u_1 = 0$ and $u_2 = 7$.

Evaluate u_5.

(c) Three cards are drawn at random from a pack of 52 cards.
Find the probability that

(i) the three cards are aces

(ii) the first card is the ace of diamonds, the second card is the ace of hearts and the third card is the ace of spades

(iii) the three cards are clubs

(iv) the three cards are from the same suit.

7. (a) Find the standard deviation of the numbers $-1,\ 3,\ 6,\ 7,\ 10$.

(b) A group consists of 5 women and 4 men.

The chairperson of the group is a woman.

The group sits in a row.

(i) How many different seating arrangements are possible?

The group decides that each man will sit between two women.

(ii) How many different seating arrangements are there?

(iii) If the chairperson is to sit in the centre, how many different seating arrangements are there?

(c) Two dice are thrown and their score recorded.

What is the probability that

(i) the combined score is 2

(ii) the combined score is 10 or 12

(iii) the combined score is an odd number greater than 3

(iv) the score on each die is odd but the number 5 is not shown?

SECTION B

Answer ONE question from this section.

8. (a) Use integration by parts to find $\int x^3 \log x \, dx$.

(b) $f(x) = f(0) + \dfrac{f'(0)x}{1!} + \dfrac{f''(0)x^2}{2!} + \ldots$ is the Maclaurin series.

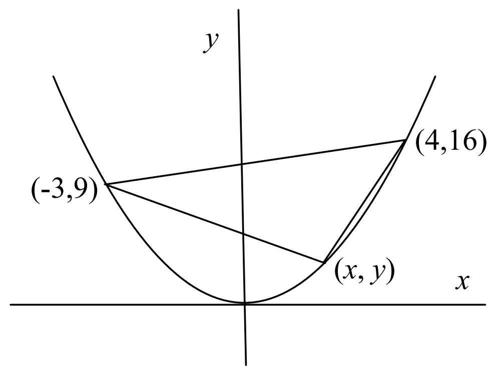
Find the first four terms of the Maclaurin series for $f(x) = \cos x$.

Write down the general term and use the Ratio Test to show that the series converges for all $x \in \mathbf{R}$.

(c) $(-3, 9)$, $(4, 16)$ and $p(x, y)$ are three points on the curve $y = x^2$.

The three points form a triangle as shown.

Find the co-ordinates of p such that the area enclosed by the triangle is a maximum.



9. (a) A dartboard is numbered from 1 to 20. Three darts are thrown at random so that each number has an equal chance of being hit.
What is the probability of the three darts hitting the numbers 18, 19 and 20 in that order?

(b) John regularly plays golf with Joe. Historically, John has won 8 out of every 10 games he played with Joe. John and Joe agree to play three games over a weekend.

Find the probability that John

(i) wins the first and second games but loses the third game,

(ii) wins any two of the three games

(iii) wins at least two of the three games played.

(c) A company which supplies packets of cereal to a supermarket claims that the packets will have a mean mass of 500 g and a standard deviation of 18 g.

The supermarket selects 36 packets at random from a large consignment and finds that the mean mass is 505 g.

At the 5% level of significance is this result consistent with the supplier's claim.

10. (a) The binary operation $*$ is defined by $x * y = x + y + 1$.

(i) Investigate if the operation is associative.

If Z, $*$ is a group with $x * y = x + y + 1$, $x, y \in \mathbf{Z}$,

(ii) find e, the identity element of the group

(iii) find x^{-1}, the inverse of x, in terms of x.

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(b) Let $G = \{1, 2, 4, 5, 7, 8\}$.

(i) Show that G is a group under multiplication modulo 9.
You may assume associativity under multiplication in **N**.

(ii) State the order of the elements of the group.

(iii) Write down the proper subgroups of G and justify your answer.

(iv) $H, \times$ is a group where $H = \{1, \omega, \omega^2\}$ and $\omega^3 = 1$.
Investigate if $H, \times$ is isomorphic to a proper subgroup of G.

11. (a) Let f be the transformation $(x, y) \to (x', y')$ where $x' = 5x$ and $y' = 3y$.

Show that the image of the circle $x^2 + y^2 = 4$ under f is an ellipse and find its eccentricity.

(b) The tangents from an external point p to a circle S touch the circle at q and r.
Prove that qr is the polar of p.

(c) f is a similarity transformation.

The image of the line segment $[pq]$ under f is the line segment $[p'q']$.

If the line M is the perpendicular bisector of $[pq]$, prove that $f(M)$ is the perpendicular bisector of $[p'q']$.