Cork Institute of Technology

Special Mathematics Examination for Engineering Degree Entry

June 2007

Time: 2 hours and 30 minutes

PAPER 2 (300 marks)

Attempt **FIVE** questions from Section **A** and **ONE** question from Section **B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

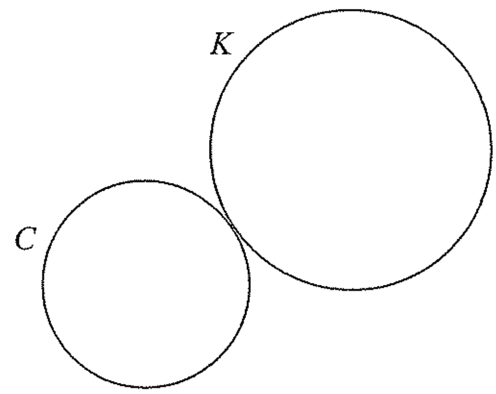
SECTION A
Answer FIVE questions from this section

1. (a) The parametric equations

$$x = 3 + 2\cos\theta, \qquad y = -4 + 2\sin\theta$$

define a circle.

Find the Cartesian equation of the circle.



(b) Circles C and K touch externally.
Circle C has equation $x^2 + y^2 - 6x + 2y - 6 = 0$.
Circle K has centre (9, 7).

(i) Find the centre and radius of C.

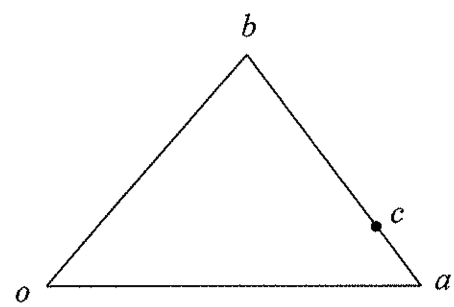
(ii) Find the equation of K

(c) Circle S has radius 2.

The centre of S is a point of the line $2x - y + 3 = 0$.

The line $x = 5$ is a tangent to S.

Find the two possible equations of S.



2. (a) oab is a triangle, where o is the origin.
c is a point on $[ab]$ such that $|ac|:|cb| = 1:3$.

(i) Express $\overrightarrow{ab}$ in terms of $\vec{a}$ and $\vec{b}$.

(ii) Express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.

(b) $\vec{u} = 4\vec{i} - 3\vec{j}$ and $\vec{v} = 7\vec{i} + k\vec{j}$.

(i) Express $\overrightarrow{uv}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Given that $\vec{u} \perp \overrightarrow{uv}$, find the value of k.

(iii) Calculate $|\overrightarrow{uv}|$.

(iv) Show that the triangle ouv is isosceles.

(v) Given that $ouvw$ is a square, find $\vec{w}$ in terms of $\vec{i}$ and $\vec{j}$.

(vi) $\vec{q}$ is the unit vector in the direction of $\vec{u}$. Express $\vec{q}$ in terms of $\vec{i}$ and $\vec{j}$.

(vii) Show that $\vec{q}^{\perp}$ is the unit vector in the direction of $\vec{w}$.

3. (a) $a(7, 3)$ and $b(-1, -5)$ are two points.

(i) Find the co-ordinates of c, the midpoint of $[ab]$.

(ii) $p(k^2, k)$ is a point such that $cp \perp ab$.
Find the co-ordinates of the two possible positions of p.

(iii) Find the area of the triangle acp, where p is in the fourth quadrant.

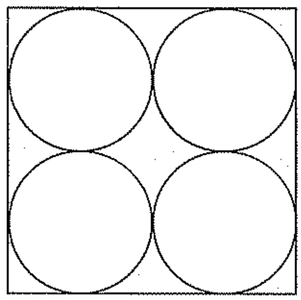
(b) $t(-1, 1)$ is a point **not** on the line $L: 2x + y - 2 = 0$.

(i) Show that the distance of t from L is greater than the distance of t from the y-axis.

(ii) Find the equations of the lines through t which make an angle of $45°$ with L.

4. (a) A is an acute angle such that $\tan A = \frac{5}{12}$.
Find $\sin 2A$, without evaluating A.

(b) The diagram shows four circles of radius 1
with each circle touching two adjacent circles.
The four circles are inscribed in a square.
Find the area of the shaded region, in terms of π.



(c) **(i)** Prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

(ii) Prove that $\cos(x - 30°) - \cos(x + 30°) = \sin x$.

5. (a) The area of an equilateral triangle is $100\sqrt{3}$ cm^2.
Find the length of a side of the triangle.

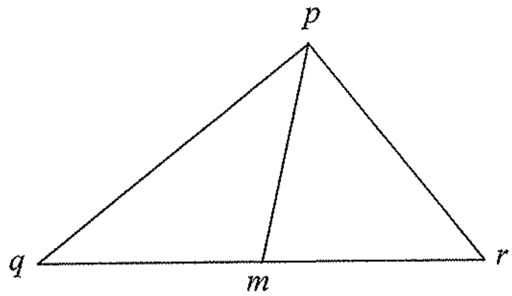
(b) Prove that $\cos^2 A + \sin^2 A = 1$.

Hence, or otherwise, solve the equation

$$\sin^2 A + 2\cos A - 2 = 0, \qquad 0 \le A \le 2\pi .$$

(c) The triangle pqr is right-angled at p.
m is the midpoint of $[qr]$.
$|pq| = 21$ and $|pr| = 20$.

Find $|\angle pmr|$, correct to the nearest degree.



6. (a) The letters of the word HISTORY are arranged at random.

(i) How many different arrangements are possible?

(ii) How many of these arrangements have the vowels together?

(b) Solve the difference equation

$$2u_{n+2} - 3u_{n+1} + u_n = 0, \quad n \ge 0$$

given that $u_0 = -1$ and $u_1 = 1$.

Show that $u_{n+1} > u_n$ for all n.

(c) One hundred books are given to a school library. The books are classified as biography, fiction or scientific and have either hardback or softback covers.
The table gives the number of each type.

	Biography	Fiction	Scientific
Hardback	15	17	34
Softback	5	18	11

A book is selected at random. Find the probability that the book selected is

(i) A hardback biography.

(ii) A hardback or a biography.

(iii) Not a softback and not scientific.

(iv) Fiction or softback but not both.

7. (a) In an election there are 12 candidates in a five seat constituency. Five of the candidates are women.

(i) In how many ways can the seats be filled?

(ii) If two women are elected, in how many ways can the seats be filled?

(b) **(i)** Show that $\frac{1}{n!} + \frac{1}{(n+1)!} = \frac{n+2}{(n+1)!}$.

(ii) A box contains 5 red, 3 white and 2 black discs.

Three discs are picked at random from the box and not replaced.

Find the probability that the 3 discs have different colours.

What is the probability that two of the discs have different colours?

(c) $\bar{x}$ and σ are, respectively, the mean and standard deviation of the set $\{x_1, x_2, x_3\}$.

(i) Show that $\frac{2\bar{x}-1}{3}$ is the mean of the set $\left\{\frac{2x_1-1}{3}, \frac{2x_2-1}{3}, \frac{2x_3-1}{3}\right\}$.

(ii) Find the standard deviation of this set in terms of σ.

SECTION B

Answer ONE question from this section.

8. (a) Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{n+2}{3^n}$ is convergent.

(b) Evaluate $\int_0^1 x^2 e^x \, dx$.

(c) An open rectangular tank is constructed of thin steel sheets. The tank has a square base of side x metres and a capacity of 32 m^3.

(i) Write an expression for the surface area of the tank in terms of x.

(ii) Find the dimensions of the tank if the surface area is a minimum.

(iii) Find the minimum surface area.

9. (a) Fifteen cards are numbered 1 to 15, each card having a unique number.
Three cards are picked at random.
Find the probability that 7 is the median number picked.

(b) A coin is biased so that a tail is 3 times more likely to occur as a head.
The coin is tossed repeatedly.

Find the probability that

(i) the first head will appear on the fourth toss of the coin

(ii) there will be exactly 2 heads in the first 8 tosses.

(c) The mean height of the population of a large city was found to be 155 cm and the standard deviation was 16 cm. A random sample of 100 was taken.
Find the probability that the mean was greater than 157 cm.

10. (a) $a = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix}$, $b = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix}$, $c = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,1\,4 \end{pmatrix}$ are permutations of the elements 1, 2, 3, 4.

(i) Simplify $a \circ b \circ c$, where $\circ$ means "after".

(ii) Find the permutation x such that $a^{-1} \circ x \circ a = c$.

(b) G is a group under multiplication, with identity e, such that for a, b elements of G

$$a^2 = e, \quad b^2 = e \quad \text{and} \quad (ab)^2 = e.$$

Prove that

(i) $a = a^{-1}$

(ii) $ab = ba$.

(c) $G = \{I, R_1, R_2, R_3\}$ is the set of rotational symmetries of a square where I, R_1, R_2, R_3 are rotations through $0°, 90°, 180°$ and $270°$ respectively.

(i) Construct a Cayley table for the composition of these rotations and verify that $G, \circ$ is a group.

(ii) H is the group $\{1, 3, 7, 9\}$, $\times$ mod 10.
Verify that $G, \circ$ is isomorphic to $H, \times$.

11. (a) An ellipse has eccentricity $\frac{1}{2}$ and the length of its major axis is 4. Find its equation in standard form.

(b) Show that under a transformation of the form

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

the midpoint of a line segment is mapped on to the midpoint of the image line segment.

(c) Given a unit circle C centre the origin, show that there is a transformation f of the type in part (b) above such that $f(C) = E$, where E is an ellipse with equation in standard form.

Prove that the locus of midpoints of parallel chords of E is a diameter of E.