1. (a) When a force of 100 kN is applied to an object of length $L_0$, with a cross sectional area of 1 cm$^2$, made out of an alloy with a Young's Modulus of $E$ GPa, the length is transformed to $L_1$. In order to measure the Young's Modulus, $E$, of a particular alloy, the following recordings were made

| $L_0$/m | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $L_1$/m | 2.01 | 4.02 | 6.03 | 8.05 | 10.06 |

The formula governing the relationship between $L_1$ and $L_0$ can be written as

$$L_1 = \left(1 + \frac{1}{E}\right) L_0 + \varepsilon,$$

where $\varepsilon$ is a small corrective constant to account for measurement error and non-linear behavior.

i. Write the above relationship in the form $y = mx + c$, clearly identifying the constants $m$ and $c$ and the variables $y$ and $x$.

[2 Marks]

ii. Find the best values of the constants $m$ and $c$ in the *least squares sense.* Make calculations correct to four significant figures.

[HINT: $\sum L_0 = 30$, $\sum L_1 = 30.17$, $\sum L_0 L_1 = 221.3$, $\sum L_0^2 = 220$.]

[6 Marks]

iii. Hence calculate the Young's Modulus of the alloy.

[3 Marks]

(b) In order to predict the number of blows $N$ required to pile drive to a depth of $D$ m in a given soil, the following recordings were made

| depth, $D$/ m | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| blows, $N$ | 10 | 25 | 65 | 155 | 390 |

i. Using three suitable data points, use *Lagrange interpolation* to provide a rough estimate for the number of blows required to reach a depth of 2.5 m. [3 Marks]

A more sophisticated method of interpolating $N(2.5)$ is to model the relationship between $N$ and $D$ using a function such as

$$N = ae^{bD}.$$

ii. Write this relationship in the form $y = mx + c$, identifying $y$, $m$, $x$, and $c$. [3 Marks]

iii. Find the best values of the constants $a$ and $b$ in the (log-linear) *least squares sense*. Make calculations correct to four significant figures. [6 Marks]

iv. Hence estimate the number of blows required to reach a depth of 2.5 m. [2 Marks]

2. (a) A light beam of span 6 m is simply supported at its endpoints. At the point $x = 1$ m there is a point load of 36 kN. Between the points $x = 2$ m and $x = 4$ m there is a U.D.L. of 72 kN m$^{-1}$.

i. Either by solving the differential equation

$$\frac{d^2M}{dx^2} = -w(x),$$

where $w(x)$ is the loading, or otherwise, find the bending moment $M(x)$. [4 Marks]

ii. The deflection, $y(x)$, at any point on the beam is found by solving the differential equation

$$EI\frac{d^2y}{dx^2} = M(x).$$

Solve the differential equation for $y(x)$. [9 Marks]

iii. At $x_1 \approx 2.959$ m we have $y'(x_1) = 0$. What can we conclude about the deflection at $x_1$? [2 Marks]

(b) Find, in terms of $EI$, the maximum deflection of a light cantilever beam of span 6 m with a constant U.D.L. of 12 kN $m^{-1}$ across the entire beam.

The deflection, $y(x)$, is the solution the fourth order differential equation

$$EI \cdot \frac{d^4y}{dx^4} = -w(x).$$

[10 Marks]

3. A quality consultant is hired to analyse the manufacturing standards at a factory.

(a) Over a period of time she finds that 6% of produced items do not reach industry standards. The consultant decides to examine a sample of 30 items. Find the probability that:

i. four items are substandard.

ii. at least two items are substandard.

[2 & 3 Marks]

(b) Substandard items are suspected to be output according to a Poisson Distribution at an average rate of 1.25 per hour.

i. Calculate the probability of the machine outputting four substandard items in a single hour.

ii. Calculate the probability of the machine outputting no substandard items over a two hour period.

[2 & 3 Marks]

(c) Last year's figures suggest that the number of substandard items produced daily are normally distributed with a mean of 10 and a standard deviation of 1.5. Assuming that the distribution of substandard items has not changed, calculate the probability that the number of substandard items is

i. greater than 14 a day.

ii. between 8 and 12 a day.

[2 & 3 Marks]

(d) The consultant ordered the foreman to record the number of substandard items produced for 50 working days. This study yielded a sample mean of 10.2 substandard items and a sample standard deviation of 1.4. Hence calculate a 95% confidence interval for the mean number of substandard items.

[4 Marks]

(e) Determine whether or not the claim that the average number of substandard items on a given day is 10 can be rejected at the 0.05 level of significance. Specify the null and alternative hypothesis.

[6 Marks]

4. (a) Suppose that the loading on a beam of span 6 m is given by

$$w(x) = 10\cos\left(\frac{(x-3)^2}{2\pi}\right).$$

Then the shear, $V(x)$, is given as the solution of the initial value problem

$$\frac{dV}{dx} = -10\cos\left(\frac{(x-3)^2}{2\pi}\right), \quad V(0) = 24.4.$$

Use a numerical method with a step size of $h = 0.1$ to estimate the value of $V(0.2)$, the shear at $x = 20$ cm.

For those using the Three Term Taylor Method, you may use

$$\frac{d}{dx}\left(-10\cos\left(\frac{(x-3)^2}{2\pi}\right)\right) = +\frac{(x-3)}{\pi}\sin\left(\frac{(x-3)^2}{2\pi}\right).$$

[HINT: Use RADIANS not degrees. Use four significant figures and/or three decimal places for intermediate calculations.]

[6 Marks]

(b) Suppose that for a load of $P$ applied to a beam of length $L$, Young's Modulus $E$ and second moment of area $I$, that the deflection of the beam is given by

$$\delta_{\text{max}} = \frac{1-\cos(kL)}{\cos(kL)}, \text{ where } k = \sqrt{\frac{P}{EI}}.$$

i. Show that

$$\delta_{\text{max}} = \sec(kL) - 1.$$

[1 Mark]

ii. Show carefully that the first two non-zero terms of the Maclaurin series for $f(x) = \sec(x)$ are given by:

$$f(x) = 1 + \frac{1}{2}x^2 + \cdots.$$

[5 Marks]

iii. If $kL \ll 1$, then $\sec(kL)$ can be approximated using the first two non-zero terms of its Maclaurin Series:

$$\sec x \approx 1 + \frac{1}{2}x^2.$$

Making this assumption, show that

$$\delta_{\text{max}} = \sec(kL) - 1 \approx \frac{PL^2}{2EI}.$$

[4 Marks]

(c) If square beams of length $L$ are manufactured from an alloy, then the maximum deflection under a uniform loading of $W$ kN m$^{-1}$ in millimetres is given by

$$\delta = \frac{WL^4}{25920}.$$

Use differentials to estimate the error in $\delta$, $\Delta\delta$ where the loading was measured to be 40 kN m$^{-1}$ and the length was measured to be 6 m with maximum errors of 0.1 kN m$^{-1}$ and 0.001 m.

Present your answer in the form

$$\delta = \delta_0 \pm \Delta\delta.$$

[9 Marks]