CORK INSTITUTE OF TECHNOLOGY
INSTITIÚID TEICNEOLAÍOCHTA CHORCAÍ

Autumn Examinations 2017/18

Module Title: Mathematics for Control and Quality

Module Code:	MATH8005
School:	Mechanical, Electrical and Process Engineering Science and Informatics
Programme Title(s):	BSc (Hons) in Advanced Manufacturing Technology BEng (Hons) in Building Energy Systems BSc (Hons) in Instrument Engineering BEng (Hons) Mechanical Engineering Systems BSc (Hons) in Process Plant Technology
Programmes Code(s):	EAMTE_8_Y4 EAMTN_8_Y4 EBENS_8_Y4 SINEN_8_Y3 EPPTE_8_Y4 EPPTN_8_Y4
External Examiner(s):	Dr Ann O Shea
Internal Examiner(s):	Dr Shane O Rourke
Instructions:	Answer ANY THREE questions. All questions carry equal marks.
Duration:	2 hours
Sitting:	Autumn 2018
Requirements for this examination:	Formulae and tables booklet (State Examinations Commission) Statistical Tables (Murdoch and Barnes)

Note to Candidates: Please check the Programme Title and the Module Title to ensure that you have received the correct examination. If in doubt please contact an Invigilator.

Q1. (a) Consider the function

$$f(t) = \begin{cases} 0 & 0 \le t < 4 \\ t-5 & t \ge 4. \end{cases}$$

(i). Sketch the graph of f. **(3 marks)**

(ii). Write $f(t)$ in terms of unit step functions. **(3 marks)**

(iii). Find the Laplace transform of f. **(3 marks)**

(b) (i). Sketch the graph of the function $f(t) = 2\delta(t) + 3\delta(t-2) - \delta(t-4)$ and find its Laplace transform $F(s)$.
(**Note:** $\delta(t-a)$ denotes the Dirac delta function.) **(3 marks)**

(ii). Find the inverse Laplace transform of $G(s) = \dfrac{2e^{-s}}{s^2} - \dfrac{e^{-5s}}{s-1}$. **(3 marks)**

(c) Solve the initial value problem

$$\frac{dx}{dt} + x = \delta(t-5), \quad x(0) = -1$$

using Laplace transforms. **(10 marks)**

Q2. (a) Let $F(z) = \mathcal{Z}[f(n)]$ denote the Z transform of f. Use the definition of the Z transform to show that

$$\mathcal{Z}[5^n] = \frac{z}{z-5}.$$

Find also the radius of convergence of $F(z)$.

(5 marks)

(b) Suppose that a drug is administered to a patient and let $y(n)$ be the number of milligrams of the drug remaining in a person's system after n hours. It is known that 40% of the amount of the drug in a person's system at a given time is in the person's system after one hour. (So if, for example, there were $10mg$ in the person's system now there would be $4mg$ in one hour's time.) The drug's instructions state that $15mg$ of the drug should be taken when required. Let $y(n)$ be the number of milligrams of the drug remaining in the person's system n hours after taking the drug.

(i). Represent the information above as a recurrence relation. **(2 marks)**

(ii). Use the recurrence relation to find $x(1)$ and $x(2)$. **(2 marks)**

(iii). Use Z transforms to find a direct formula for $y(n)$. **(5 marks)**

(iv). For all practical purposes there is no trace of the drug remaining in a person's system once $y(n)$ is less than 10^{-9}. Find the smallest value of n for which this is the case. **(3 marks)**

(c) Use $\mathcal{Z}$-transforms to solve the recurrence relation

$$\begin{aligned} x_{n+1} &= 5x_n - 2^n \\ x_0 &= 1. \end{aligned}$$

(8 marks)

Q3. (a) In a factory, machines A, B and C produce 20%, 35% and 45% respectively of the total output. Of their respective outputs, 3%, 2% and 1% are defective.

(i). Find the probability that a randomly chosen item from the factory's output is defective. **(4 marks)**

(ii). An item is chosen at random from the defectives. Find the probability that it was produced on machine A. **(5 marks)**

(b) Suppose that 8% of the tyres produced on a production line are defective. Find the probability that exactly one tyre is defective, using

(i). the Binomial distribution; **(3 marks)**

(ii). the Poisson approximation to the Binomial distribution. **(3 marks)**

(c) A firm is to introduce an acceptance sampling scheme as follows.

Take a sample of 40 items and accept the batch if at most one non-conforming item is found. Otherwise reject the batch.

Find the probability of acceptance of this plan if batches are received containing

(i). 1% non-conforming items; **(4 marks)**

(ii). 5% non-conforming items. **(4 marks)**

Sketch the Operating Characteristic curve for this acceptance sampling scheme. **(2 marks)**

Q4. (a) Sugar is filled in packets in a food packaging plant. The weights of the bags are normally distributed with a mean of 1kg and a standard deviation of 2g.

(i). Find the probability that a random packet of sugar weighs between 995g and 998g. **(4 marks)**

(ii). Under EU law a minimum weight must be printed on each bag and not more than 1% of bags may weigh less than this printed weight. What value should this printed weight be? **(5 marks)**

(b) Drink is dispensed in bottles on a filling line. A sample of 28 bottles is taken and the mean volume is found to be 498ml with a standard deviation of 2.5ml. Establish a 99% confidence interval for the mean volume of drink in all bottles on the filling line. **(7 marks)**

(c) Samples of light bulbs of each of two brands were taken and their respective lifetimes (in hours) was noted. The following data summarises the information found.

Brand A	Brand B
$n_1 = 15$	$n_2 = 25$
$\bar{x}_1 = 1035$	$\bar{x}_2 = 1098$
$s_1 = 112$	$s_2 = 125$

Is there statistically significant evidence that Brand B has a longer lifetime than Brand A? Use a 5% level of significance.
Note: You may assume that the population standard deviations for the two types of light bulb are equal. **(9 marks)**

—Table of Laplace Transforms—

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

$F(s)$	$f(t)$
$\frac{a}{s}$	a
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{n!}{s^{n+1}}$	t^n
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2+a^2}$	$\sin at$
$F(s+a)$	$e^{-at}f(t)$

[P.T.O.]

—Table of Laplace Transforms—(continued)

$F(s)$	$f(t)$
$\frac{1}{s}$	$u(t)$
$\frac{e^{-cs}}{s}$	$u(t-c)$
$e^{-cs}F(s)$	$f(t-c)u(t-c)$
1	$\delta(t)$
e^{-as}	$\delta(t-a)$
$e^{-as}f(a)$	$f(t)\delta(t-a)$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$sF(s)-f(0)$	$f'(t)$
$s^2F(s)-sf(0)-f'(0)$	$f''(t)$

—Table of $\mathcal{Z}$- Transforms—

$$f(n) \leftrightarrow F(z) = \mathcal{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

$f(n)$	$F(z)$
$\delta(n)$	1
a^n or $a^n u(n)$	$\dfrac{z}{z-a}$
$u(n)$ or 1 or 1^n	$\dfrac{z}{z-1}$
na^{n-1}	$\dfrac{z}{(z-a)^2}$
n	$\dfrac{z}{(z-1)^2}$
$n(n-1)$	$\dfrac{2z}{(z-1)^3}$
n^2	$\dfrac{z^2+z}{(z-1)^3}$
$f(n+1)$	$zF(z) - zf(0)$
$f(n+2)$	$z^2F(z) - z^2f(0) - zf(1)$
$f(n-m)u(n-m)$	$z^{-m}F(z)$

Formulae — Statistics and Probability

Probability

Addition Law $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Multiplication Law $P(A \text{ and } B) = P(B|A)P(A)$

Mean and Variance of a Discrete Probability Distribution

$$E(X) = \sum_i x_i P(x_i) \qquad V(X) = E(X^2) - (E(X))^2$$

Binomial Distribution $P(X = r) = C(n, r)p^r(1-p)^{n-r}$

Poisson Distribution $P(X = r) = \dfrac{e^{-m}m^r}{r!}$

Hypergeometric Distribution $P(X = r) = \dfrac{C(M, r)\, C(N - M, n - r)}{C(N, n)}$

Sampling Theory

Sample Mean $\bar{x} = \dfrac{\sum x}{n}$

Sample Standard Deviation $s = \sqrt{\dfrac{\sum (x - \bar{x})^2}{n - 1}} = \sqrt{\dfrac{\sum x^2 - \frac{(\sum x)^2}{n}}{n - 1}}$

Sampling Distribution of the Mean $E(\bar{X}) = \mu \qquad SE(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$

Normal Distribution

$$Z = \frac{x - \mu}{\sigma} \text{ where } X \text{ is } N(\mu, \sigma); \qquad Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \text{ where } \bar{X} \text{ is } N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Statistical Inference

Estimation

$$\bar{x} \pm z\frac{\sigma}{\sqrt{n}} \quad ; \quad \bar{x} \pm z\frac{s}{\sqrt{n}} \quad ; \quad \bar{x} \pm t_{n-1}\frac{s}{\sqrt{n}}$$

$$\frac{(n-1)s^2}{\chi^2_{hi}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{lo}}$$

Hypothesis Testing

One sample tests

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \quad ; \quad z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \quad ; \quad t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

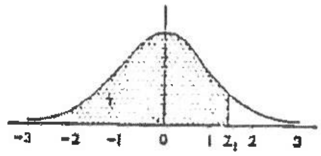
Two sample tests

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$t_{n_1+n_2-2} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad \text{where } s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

[illegible]

Area under the Normal Curve



z	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
0·0	0·5000	5040	5080	5120	5160	5199	5239	5279	5319	5359
0·1	0·5398	5438	5478	5517	5557	5596	5636	5675	5714	5753
0·2	0·5793	5832	5871	5910	5948	5987	6026	6064	6103	6141
0·3	0·6179	6217	6255	6293	6331	6368	6406	6443	6480	6517
0·4	0·6554	6591	6628	6664	6700	6736	6772	6808	6844	6879
0·5	0·6915	6950	6985	7019	7054	7088	7123	7157	7190	7224
0·6	0·7257	7291	7324	7357	7389	7422	7454	7486	7517	7549
0·7	0·7580	7611	7642	7673	7704	7734	7764	7794	7823	7852
0·8	0·7881	7910	7939	7967	7995	8023	8051	8078	8106	8133
0·9	0·8159	8186	8212	8238	8264	8289	8315	8340	8365	8389
1·0	0·8413	8438	8461	8485	8508	8531	8554	8577	8599	8621
1·1	0·8643	8665	8686	8708	8729	8749	8770	8790	8810	8830
1·2	0·8849	8869	8888	8907	8925	8944	8962	8980	8997	9015
1·3	0·9032	9049	9066	9082	9099	9115	9131	9147	9162	9177
1·4	0·9192	9207	9222	9236	9251	9265	9279	9292	9306	9319
1·5	0·9332	9345	9357	9370	9382	9394	9406	9418	9429	9441
1·6	0·9452	9463	9474	9484	9495	9505	9515	9525	9535	9545
1·7	0·9554	9564	9573	9582	9591	9599	9608	9616	9625	9633
1·8	0·9641	9649	9656	9664	9671	9678	9686	9693	9699	9706
1·9	0·9713	9719	9726	9732	9738	9744	9750	9756	9761	9767
2·0	0·9772	9778	9783	9788	9793	9798	9803	9808	9812	9817
2·1	0·9821	9826	9830	9834	9838	9842	9846	9850	9854	9857
2·2	0·9861	9864	9868	9871	9875	9878	9881	9884	9887	9890
2·3	0·9893	9896	9898	9901	9904	9906	9909	9911	9913	9916
2·4	0·9918	9920	9922	9925	9927	9929	9931	9932	9934	9936
2·5	0·99379	99396	99413	99430	99446	99461	99477	99492	99506	99520
2·6	0·99534	99547	99560	99573	99585	99598	99609	99621	99632	99643
2·7	0·99653	99664	99674	99683	99693	99702	99711	99720	99728	99736
2·8	0·99744	99752	99760	99767	99774	99781	99788	99795	99801	99807
2·9	0·99813	99819	99825	99831	99836	99841	99846	99851	99856	99861
3·0	0·99865	99869	99874	99878	99882	99886	99889	99893	99897	99900
3·1	0·99903	99906	99910	99913	99916	99918	99921	99924	99926	99929
3·2	0·99931	99934	99936	99938	99940	99942	99944	99946	99948	99950
3·3	0·99952	99953	99955	99957	99958	99960	99961	99962	99964	99965
3·4	0·99966	99968	99969	99970	99971	99972	99973	99974	99975	99976
3·5	0·99977	99978	99978	99979	99980	99981	99981	99982	99983	99983
3·6	0·99984	99985	99985	99986	99986	99987	99987	99988	99988	99989
3·7	0·99989	99990	99990	99990	99991	99991	99992	99992	99992	99992
3·8	0·99993	99993	99993	99994	99994	99994	99994	99995	99995	99995
3·9	0·99995	99995	99996	99996	99996	99996	99996	99996	99997	99997

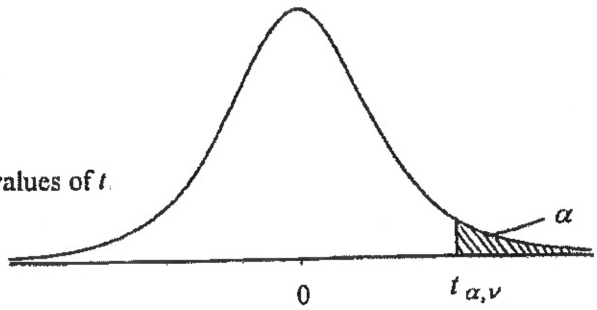
Table 7 Percentage Points of the *t* Distribution

The table gives the value of $t_{\alpha,\nu}$ – the 100α percentage point of the t distribution for ν degrees of freedom.

The values of t are obtained by solution of the equation:

$$\alpha = \Gamma[\tfrac{1}{2}(\nu+1)][\Gamma(\tfrac{1}{2}\nu)]^{-1}(\nu\pi)^{-1/2}\int_t^{\infty}(1+x^2/\nu)^{-(\nu+1)/2}\,dx$$

Note: The tabulation is for one tail only, that is, for positive values of t. For $|t|$ the column headings for α should be doubled.



$\alpha =$	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
$\nu = 1$	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

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