CORK INSTITUTE OF TECHNOLOGY

INSTITIÚID TEICNEOLAÍOCHTA CHORCAÍ

Autumn Examinations 2017/2018

Module Title: Multivariable Calculus

Module Code: **MATH8010**

School: School of Science and Informatics
School of Mechanical, Electrical and Process Engineering

Programme Title: BEng (Hons) in Chemical & Biopharmaceutical Engineering
BEng (Hons) in Electronic Engineering
BEng (Hons) in Electrical Engineering

Programme Code: ECPEN_8_Y4
EELES_8_Y4
EELPS_8_Y4

External Examiner(s): **Dr Ann O'Shea**
Internal Examiner(s): **Dr Maryna Lishchynska**

Instructions: Answer **ALL** questions. Show all calculations in full.

Duration: 2 hours

Sitting: Autumn 2018

Requirements for this examination: Mathematical Tables

Note to Candidates: Please check the **Programme Title** and the **Module Title** to ensure that you have received the correct examination.
If in doubt please contact an Invigilator.

Q1.

a) A particle moves along a curve described parametrically by $\mathbf{r}(t) = \left[\cos(3t + \pi), \sin(3t + \pi), t\right]$ where t is time in seconds.

(i) Find $\frac{d\mathbf{r}}{dt}$ and $\left|\frac{d\mathbf{r}}{dt}\right|$. **[6 marks]**

(ii) Determine the velocity of the particle at time t=0.

[2 marks]

(iii) Find the distance the particle covers in the first 10 seconds. Assume lengths given in millimetres.

[4 marks]

b) The concentration of smoke in the air on the factory floor is given by $C(x, y, z) = \left(x^2 + y^2\right)e^{-z}$.

(i) Given that the diffusion occurs in the direction from high concentration to low concentration regions, find the direction of smoke diffusion at the point (1, 1, 0).

[5 marks]

(ii) What is the maximum rate of smoke diffusion at this point?

[2 marks]

c) Find the values of C_1, C_2 and C_3 for which the fluid flow described by $\mathbf{v} = \left[x + 4y, C_3x - 3z, C_1y - C_2x\right]$ is irrotational.

[6 marks]

P.T.O.

Q2.

a)

(i) The voltage V generated in a closed loop C by the electric field $\mathbf{E}$ is given by $V = \oint_C \mathbf{E} \bullet d\mathbf{r}$. Find the voltage generated by the electric field $\mathbf{E} = (-xy, 3y^2)$ in the closed loop comprising the <u>entire perimeter</u> of the semi-circle: $x^2 + y^2 \leq 1,\ x \geq 0$. Make a sketch and assume positive direction.

[13 marks]

(ii) Use Green's theorem to verify the answer from part (i).

[6 marks]

b) A lamina is modelled by the region R in the xy-plane bounded by $y = x^2$ and $y = 2x$.

(i) Sketch this region and use a double integral to compute the area. The unit of length is centimetre.

[6 marks]

(ii) Given that the temperature in the lamina varies according to $T(x, y) = x^2 + xy$ (in °C), use a double integral to find the mean temperature in the lamina.

[8 marks]

c) A spherical fertiliser pellet of radius 3 mm is being dissolved in water and is releasing the fertiliser at the rate $\mathbf{q} = [x, y, z]$ (in mg s^{-1} mm^{-2}).

(i) Use the spherical coordinates and compute $\mathbf{q} \bullet \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit normal vector to the surface of the sphere. Give your answer in the simplest form.

[5 marks]

(ii) Find the total flux of the fertiliser through the surface of the pellet $Q = \iint_S \mathbf{q} \bullet \hat{\mathbf{n}}\, dS$.

[6 marks]

(iii) Use Gauss's divergence theorem to verify your answer from part (ii).

[6 marks]

P.T.O.

Q3. Consider a 2 m long metal rod. The temperature $u(x,t)$ at a point x along the rod at any time t is found by solving the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ where k is the material property.

The left end of the rod (x=0) is maintained at 20°C and the right end is suddenly dipped into snow (0°C). The initial temperature distribution in the rod is given by $u(x,0)=f(x)$.

(i) Use the substitution $u(x,t) = v(x,t) + 20 - 10x$ to reduce the above problem to a homogeneous boundary value problem and then solve it.

(ii) Show that if initially the rod was at room temperature i.e. $f(x) = 20$, the particular solution of this heat conduction problem is

$$u(x,t) = 20 - 10x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{40}{n\pi} \sin\left(\frac{n\pi}{2}x\right) e^{-k\left(\frac{n\pi}{2}\right)^2 t}$$

[25 marks]

Useful Formulae

Arclength of the curve $\mathbf{r}(t) = [x(t), y(t), z(t)]$ is

$$s(t) = \int_{t_0}^{t} \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

A	0	π	2π
$\sin A$	0	0	0
$\cos A$	1	-1	1

Half-range Cosine Fourier Series	**Half-range Sine Fourier Series**
$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$ $A_0 = \frac{2}{L}\int_0^L f(x)dx \; ; A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$	$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$ $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

Trigonometric Integrals

$$\int \cos ax\, dx = \frac{1}{a}\sin ax\, ; \quad \int \sin ax\, dx = -\frac{1}{a}\cos ax$$

$$\int x \sin ax\, dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}; \quad \int x\cos ax\, dx = \frac{x \sin ax}{a} + \frac{\cos ax}{a^2}$$

$$\int (L-x)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{L(L-x)}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$\int (L-x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L(L-x)}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

Polar coordinates	Elliptic coordinates
$x = r\cos\theta,\ y = r\sin\theta$	$x = ar\cos\theta,\ y = br\sin\theta$
$\lvert J\rvert = r$	$\lvert J\rvert = abr \; ; 0 \le r \le 1$

Cylindrical coordinates	Spherical coordinates
$x = r\cos\theta,\ y = r\sin\theta,\ z = z$	$x = r\sin\varphi\cos\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\varphi$
$\lvert J\rvert = r$	$\lvert J\rvert = r^2\sin\varphi$
unit normal to the curved surface: $\hat{\mathbf{n}} = (\cos\theta, \sin\theta,\ 0)$	unit normal to the surface: $\hat{\mathbf{n}} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta,\ \cos\varphi)$