# Silence Please
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## CIT Semester 1 Examinations 2018/19

**Note to Candidates:** Check the Programme Title and the Module Description to ensure that you have received the correct examination. If in doubt please contact an Invigilator.

**Module Title:** **Mathematics for Engineers**

**Module Code:** **STAT8002**

**Programme Title(s):** BEng (Hons) Mechanical Eng Y4

BEng Hons Biomedical Eng Y4

**Block Code(s):** **EMECH_8_Y4** **EBIOM_8_Y4**

**External Examiner(s):** **Dr. Katarina Domijan**

**Internal Examiner(s):** Dr. Seán Lacey

**Instructions:** Answer ALL questions.

Questions DO NOT carry equal marks.

Towards the back of the examination paper are required formulae.

**Duration:** 2 Hours

**Required Items:** Calculator, Murdoch & Barnes Tables, Log/Formulae Tables

1. **[35 marks]**

The vibrations in an elastic string are governed by the one dimensional wave equation

$$u_{tt} = c^2 u_{xx}.$$

where $u(x,t)$ is the vertical deviation of the string from the equilibrium position and $c$ is a fixed constant equal to the propagation speed of the wave. Consider the case where the string is stretched to a length of 4 m with the two ends fixed. The string is disturbed so that its initial deviation is given by,

$$f(x) = \begin{cases} x & \text{if } \; x \leq 2, \\ 4 - x & \text{if } \; x > 2. \end{cases}$$

The string is released with an initial velocity $g\,(x) = 2$ m/s and subsequently undergoes small transverse oscillations.

1. (a) Formulate the mathematical problem modelling the oscillations in the string.
   **[5 marks]**

   (b) Use separation of variables to solve the mathematical problem and show that the oscillations are given by an expression of the form

$$u\,(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{cn\pi t}{4}\right) + B_n \sin\left(\frac{cn\pi t}{4}\right)\right] \sin\left(\frac{n\pi x}{4}\right).$$

   **[12 marks]**

   (c) Derive explicit expressions for the constants $A_n$ and $B_n$.
   **[18 marks]**

2. **[35 marks]**

A company has three production plants, each of which produces three different models of a particular product. The daily capacities (in thousands of units) of the three plants are as follows:

| | Model 1 | Model 2 | Model 3 |
|---|---|---|---|
| Plant 1 | 8 | 4 | 8 |
| Plant 2 | 6 | 6 | 3 |
| Plant 3 | 12 | 4 | 8 |

The total demand for Model 1 is 300,000 units, for Model 2 is 172,000 units and for Model 3 is 249,500 units. Moreover, the daily operating cost for Plant 1 is €55,000, for Plant 2 is €60,000 and for Plant 3 is €60,000.

(a) By letting

$$\begin{aligned} p_1 &= \#\text{ days Plant 1 is open;} \\ p_2 &= \#\text{ days Plant 2 is open;} \\ p_3 &= \#\text{ days Plant 3 is open;} \end{aligned}$$

Express this minimisation problem in its standard form - i.e., formulate the equations that describe the problem outlined above.

**[5 marks]**

(b) Hence, formulate the dual maximisation problem in its standard form and express as a simplex tableau.

**[8 marks]**

(c) From the simplex tableau in part (b):

i. State which variable will enter the basis and which variable will depart the basis;

**[2 marks]**

ii. Apply **one** iteration of the simplex method.

**[8 marks]**

(d) Application of the simplex method and reversal of the dual maximisation problem yields the following optimal table:

| Basis | $w$ | $p_1$ | $p_2$ | $p_3$ | $s_1$ | $s_2$ | $s_3$ | $RHS$ |
|---|---|---|---|---|---|---|---|---|
| $p_1$ | 0 | 1 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{12}$ | $-\frac{1}{4}$ | $\frac{45}{2}$ |
| $p_2$ | 0 | 0 | 1 | 0 | $-\frac{1}{9}$ | $\frac{2}{9}$ | 0 | $\frac{21}{2}$ |
| $p_3$ | 0 | 1 | 0 | 1 | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{19}{4}$ |
| $w$ | 1 | 0 | 0 | 0 | $\frac{5}{3}$ | $\frac{95}{12}$ | $\frac{5}{4}$ | $\frac{4305}{2}$ |

i. How many days should Plant 1, Plant 2 and Plant 3 be open in order to minimise operating cost and satisfy total demand?

**[3 marks]**

ii. What is the minimum operating cost required to meet the total demand?

**[1 mark]**

iii. Were Model 1, Model 2 and/or Model 3 over-produced? Explain your answer.

**[3 marks]**

(e) In what range can the demand for Model 1 vary without changing the optimal basis?

**[4 marks]**

(f) If the demand for Model 1 increased by 10,000 units what impact will this have on the overall cost?

**[1 mark]**

3. **[30 marks]**

Suppose that we wish to improve the yield of a polishing operation. The three inputs (factors) that are considered important to the operation are Speed ($A$), Feed ($B$), and Depth ($C$). We want to ascertain the relative importance of each of these factors on Yield ($Y$). Speed, Feed and Depth can all be varied continuously along their respective scales, from a low to a high setting. Yield is observed to vary smoothly when progressive changes are made to the inputs. This leads us to believe that the ultimate response surface for $Y$ will be smooth.

| Test | Observation |
| --- | --- |
| (1) | 106, 103, 100 |
| $A$ | 198, 210 , 204 |
| $B$ | 149, 159, 170 |
| $AB$ | 355, 340, 360 |
| $C$ | 197, 212, 200 |
| $AC$ | 243, 247, 250 |
| $BC$ | 329, 341, 340 |
| $ABC$ | 383, 370, 370 |

(a) Calculate the estimate of the $AB$ interaction effect.

**[5 marks]**

(b) Determine an estimate of the error variance.

**[8 marks]**

(c) What is the null and alternative hypotheses when testing the significance of the $AB$ interaction effect?

**[2 marks]**

(d) Test the significance of the $AB$ interaction effect, using a 5% level of significance. State the conclusion of the test in context.

**[10 marks]**

(e) Plot the $AB$ interaction effect.

**[5 mark]**

# Useful formulae

*Integration by parts:*

$$\int u dv = uv - \int v du$$

*Trigonometric integrals:*

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b); \quad \int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)$$

$$\int x \sin ax\, dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}; \quad \int x\cos ax\, dx = \frac{x \sin ax}{a} + \frac{\cos ax}{a^2}$$

$$\int (L-x)\sin\frac{n\pi\, x}{L}\,dx = -\frac{L(L-x)}{n\pi}\cos\frac{n\pi\, x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi\, x}{L}$$

$$\int (L-x)\cos\frac{n\pi\, x}{L}\,dx = \frac{L(L-x)}{n\pi}\sin\frac{n\pi\, x}{L} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi\, x}{L}$$

**Orthogonality of sine and cosine**
*Full range*

$$\begin{aligned}
\int_{-l}^{l} \sin n\pi x/l\ \sin m\pi x/l\ \ dx \quad &= 0 \qquad m \neq n \\
&= l \qquad m = n \neq 0 \\
&= 0 \qquad m = n = 0 \\
\int_{-l}^{l} \cos n\pi x/l\ \cos m\pi x/l\ \ dx \quad &= 0 \qquad m \neq n \\
&= l \qquad m = n \neq 0 \\
&= 2l \qquad m = n = 0
\end{aligned}$$

$$\int_{-l}^{l} \cos n\pi x/l\ \sin m\pi x/l\ \ dx = 0 \qquad \text{all } m, n$$

*Half range*

$$\begin{aligned}
\int_{0}^{l} \sin n\pi x/l\ \sin m\pi x/l\ \ dx \quad &= 0 \qquad m \neq n \\
&= l/2 \qquad m = n \neq 0 \\
&= 0 \qquad m = n = 0 \\
\int_{0}^{l} \cos n\pi x/l\ \cos m\pi x/l\ \ dx \quad &= 0 \qquad m \neq n \\
&= l/2 \qquad m = n \neq 0 \\
&= l \qquad m = n = 0
\end{aligned}$$

A periodic function $f(t)$ with a period of $T$ can be expanded in the *Fourier Series* (is a series of sinusoidal periodic functions) as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right]$$

where the *Fourier coefficients* are given by

| Coefficients | Half-range **cosine** series | Half-range **sine** series |
|---|---|---|
| $a_0$ | $a_0 = \frac{2}{L}\int_0^L f(t)dt$ | 0 |
| $a_n$ | $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$ | 0 |
| $b_n$ | 0 | $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$ |

When using general Fourier formulae, assume period $T=2L$, consider integral over $[0, L]$ and double the result.

**$2^k$ design, n replicates.**

Effect estimate given by $\frac{(Contrast)}{n.2^{k-1}}$

Effect SS given by $\frac{(Contrast)^2}{n.2^k}$

$V(\text{effect estimate}) = \frac{\sigma_e^2}{n.2^{k-2}}$, where $\sigma_e^2$ is the error variance .

$$SS_T = \sum (X_{ij} - \bar{X})^2,$$
$$SS_B = n\sum (\bar{X}_j - \bar{X})^2,$$
$$SS_W = \sum (X_{ij} - \bar{X}_j)^2.$$