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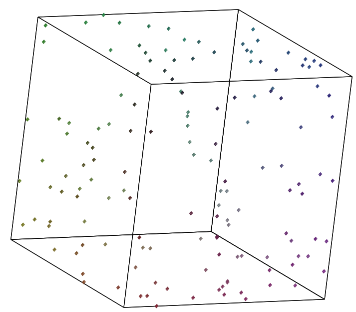
CIT Semester 2 Examinations 2018/19

Note to Candidates:	Check the Programme Title and the Module Description to ensure that you have received the correct examination. If in doubt please contact an Invigilator.
Module Title:	**Technological Mathematics 312**
Module Code:	**MATH7021**
Programme Title(s):	BEng in Civil Engineering Y3 BEng Environmental Eng Y3
Block Code(s):	**CCIVL_7_Y3** **CENVI_7_Y3**
External Examiner(s):	Prof. Brien Nolan
Internal Examiner(s):	Dr. J.P. Mc Carthy
Instructions:	Answer ALL Questions. Please find helpful formulae and tables at the back of this exam paper.
Duration:	2 hours
Required Items:	Calculator, Log/Formulae Tables

1. (a) In *3D Laser Scanning*, an engineer can generate a set of coordinates on the surface of an existing structure. For example, after scanning a box-shaped component, she might have the following:



Like any measurement, the coordinates contain errors, but the engineer can take points that lie on a plane, and find, in the *Least Squares* sense, the *plane* of best fit: $z = ax + by + c$. The *normal equations* will be given by three equations in three unknowns.

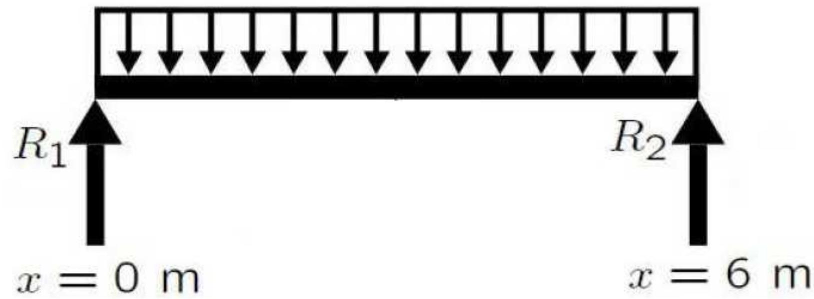
Suppose the normal equations are given by:

$$\begin{aligned} a - 2b + c &= 8 \\ 2a - b + 2c &= 10 \\ 3a - b + 2c &= 11 \end{aligned}$$

Use *Gaussian Elimination* to find the values of a, b, c.

[7 Marks]

(b) Suppose that there is a uniformly distributed load of w kN m^{-1} across a beam of length 6 m that is simply supported at $x = 0$ and $x = 6$ m with reactions R_1 and R_2 as shown:



As the beam is in equilibrium, the nett vertical force and the nett moment about $x = 0$ are both zero so that the following hold:

$$\begin{aligned} R_1 + R_2 - 6w &= 0 \\ 0R_1 + 6R_2 - 18w &= 0 \end{aligned}$$

i. Find the solution of the linear system in terms of a parameter t.

[4 Marks]

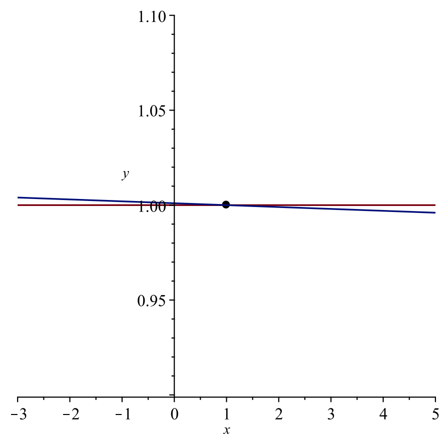
ii. Find the solution of the linear system if $w = 10$.

[1 Mark]

(c) As plotted below, the lines

$$0.001x + 1y = 1.001$$
$$0x + 1y = 1$$

have unique intersection $(1, 1)$.



i. If the 0.001 is rounded to 0, and the 1.001 rounded to 1, then the linear system is

$$0x + 1y = 1$$
$$0x + 1y = 1$$

A. Use *Gaussian Elimination* to find the solution of this linear system. [4 Marks]

B. Explain this result geometrically. [1 Mark]

ii. If the 0.001 is rounded to 0, then the linear system is

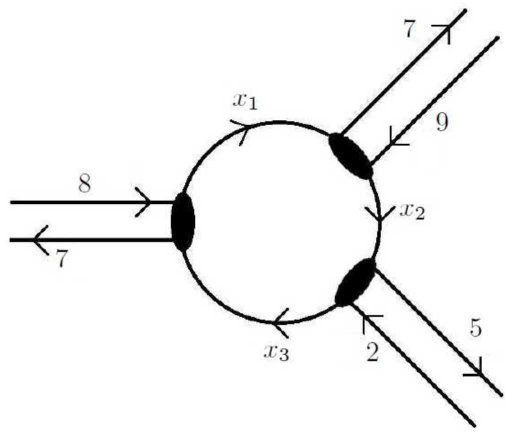
$$0x + 1y = 1$$
$$0x + 1y = 1.001$$

A. Use *Gaussian Elimination* to find the solution of this linear system. [3 Marks]

B. Explain this result geometrically. [1 Mark]

iii. Hence, comment briefly on rounding in the context of linear systems. [1 Mark]

(d) Consider a roundabout as given by the following traffic flow diagram.



i. Write down the linear system governing the flow.

[2 Marks]

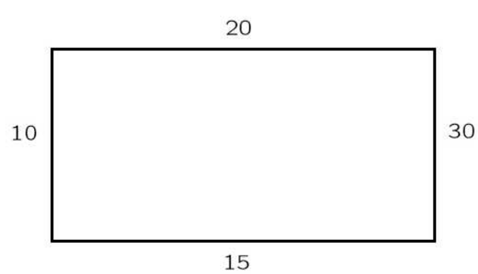
ii. Solve the linear system using *Gaussian Elimination.*

[4 Marks]

iii. Given that cars approaching the roundabout must turn left, and the model doesn't include queuing, is it possible that $x_1 = 7$? Justify your answer while considering the traffic flow diagram.

[1 Mark]

(e) The following plate has dimensions 4:2 and is subject to boundary temperatures as shown:



i. Using an appropriate grid, find a linear system whose solution approximates the heat distribution of the plate at internal grid-points using the *Mean-Value Property.* Assume the boundary temperatures have units of °C.

[3 Marks]

ii. Starting at the initial temperature of $\mathbf{T}^0 = (T_1^0, T_2^0, T_3^0) = (15, 18, 22)$ °C, use two iterations of the *Jacobi Method* to approximate the solution of the linear system. Use **three significant figures** for all calculations.

[3 Marks]

2. (a) Find, using **only** the method of undetermined coefficients, the general solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = 12.$$

This differential equation models the displacement after t seconds, $y(t)$, of a damped harmonic oscillator, subject to a constant external force.

[7 Marks]

(b) Solve, using **only** the method of undetermined coefficients, the initial value problem

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x(t) = 2t\,; \qquad x(0) = 0\,,\; x'(0) = 0.$$

This differential equation models the displacement after t seconds, $x(t)$, of a damped harmonic oscillator, initially at rest in the equilibrium position, and subject to a linearly increasing external force.

[8 Marks]

3. (a) A semi-circular disk $\mathcal{R}$ is described by $x^2 + y^2 \leq 4$; $y \geq 0$. Assume SI units.

i. *Roughly* sketch the region.

[1 Mark]

ii. Where $(\overline{x}, \overline{y})$ is the centroid, write down $\overline{x}$.

[1 Mark]

iii. Calculate

$$\overline{y} = \frac{\iint_{\mathcal{R}} y\,dA}{A}.$$

[5 Marks]

iv. Indicate on your sketch the position of the centroid.

[1 Mark]

(b) A box $\square$ is given by

$$[0, 2] \times [0, 3] \times [0, 4].$$

For this box, calculate the mass

$$M = \iiint_{\square} \rho(x, y, z)\,dV,$$

where $\rho(x, y, z) = 2xy^2z^3$. Assume SI units.

[7 Marks]

4. (a) Assuming linear air drag, the speed of an object, $v(t)$, t seconds after being dropped from a height, measured in metres, is given by the solution of the differential equation:

$$\frac{dv}{dt} = 9.8 - 0.2\,v(t); \qquad v(0) = 0.$$

i. Solve the differential equation using **only** *Laplace transforms.*

[5 Marks]

ii. Hence, or otherwise, find the *terminal velocity* v_∞ [HINT: for $t \to \infty$ *large*, $v(t) \approx v_\infty$]

[2 Marks]

(b) The differential equation governing the displacement $y(t)$ of a *damped harmonic oscillator* is given by:

$$y''(t) + 11\,y'(t) + 30\,y(t) = 0\,; \qquad y(0) = 2\,,\;\; y'(0) = 0.$$

i. Solve the differential equation using **only** *Laplace transforms.*

[9 Marks]

ii. Is the oscillator underdamped, overdamped, or critically damped?

[1 Mark]

(c) Consider a light, simply supported beam of span 6 m under a *quadratic* load, equal to zero at the end points and rising to a maximum of 54 kN m^{-1} at the midpoint. The *bending moment*, $m(x)$, is given as the solution of the ordinary differential equation

$$\frac{d^2m}{dx^2} = 6x^2 - 36x\,; \qquad m(0) = 0,\;\; m'(0) = 108.$$

i. Solve the differential equation using **only** *Laplace transforms.*

[6 Marks]

ii. Hence, given that the *shear force*, $v(x)$, is given by

$$v(x) = \frac{dm}{dx},$$

calculate the shear force at $x = 3$ m.

[2 Marks]

iii. What does the answer to part ii. suggest about the *location* of the maximum bending moment?

[1 Mark]

(d) Solve the following system of differential equations using *Laplace Transforms*:

$$\frac{dx}{dt} = -\frac{1}{2}x(t)\,; \qquad x(0) = 2$$

$$\frac{dy}{dt} = \frac{1}{2}x(t) - \frac{1}{4}y(t)\,; \qquad y(0) = 4$$

This system of differential equations models the salt contents, $x(t)$ and $y(t)$, of two vats of brine in cascade, of volumes 20 and 40 litres, t minutes after a salt free source starts flowing into A at a rate of 10 litres per minute. The first vat has an initial salt content of 2 kg and the second an initial salt content of 4 kg.

[9 Marks]

Undetermined Coefficients

If the auxiliary equation to a damped harmonic oscillator has roots $p \pm iq$, then

$$y_H(t) = e^{pt}(A\cos(qt) + B\sin(qt)).$$

Laplace Transforms Table

The Laplace Transform is defined by

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}\,dt.$$

$f(t)$	$F(s)$
$A =$ constant	$\frac{A}{s}$
t^N	$\frac{N!}{s^{N+1}}$
$\frac{t^{N-1}}{(N-1)!}$	$\frac{1}{s^N}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$f(t)e^{-at}$	$F(s+a)$
$f'(t)$	$s \cdot F(s) - f(0)$
$f''(t)$	$s^2 \cdot F(s) - s \cdot f(0) - f'(0)$